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A study of synchrotron radiation near the orbit

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Abstract. We study the radiation intensity produced by a monoenergetic electron in circular motion as a function of the distance to the orbit. The analysis is carried out along two directions, namely in the radial direction, and along the tangent to the orbit. We find that, near the orbit, the intensity of radiation in the radial direction differs significantly from the value given in Schott's formula. Together with an increase of the radiation intensity as we move closer to the orbit, the angular spread of the radiation decreases. As a result, a focusing effect of the radiation in the orbit plane is obtained. In particular, when radiation is detected almost touching the electron orbit, all the electron radiation tends to be concentrated into a line, giving rise to a very high density of energy per unit area. Except for points very close to the tangential point, we show that the intensity of radiation in the forward direction is given by Schwinger's formula, independent of the distance to the tangential point.

1. Introduction

The radiation emitted along the radial direction by a monoenergetic electron in a circular orbit was studied a long time ago by Schott (1912), who arrived at the following expression for the spectral distribution of the radiation produced during a period of motion:

$$\frac{\mathrm{d}P_n}{\mathrm{d}\Omega'}=\frac{e^2\omega^4a^2n^2}{2\pi c^3\beta^2}\left[\beta^2\left(\frac{\mathrm{d}J_n(n\beta\sin\theta)}{\mathrm{d}(n\beta\sin\theta)}\right)^2+\cot^2\theta J_n^2(n\beta\sin\theta)\right] \tag{1.1}$$

where a is the orbit radius, ω denotes the electron angular frequency, $\beta=a\omega/c$ and θ is the usual spherical polar angle in a system of coordinates with the origin at the centre of the orbit, which lies in the XY plane. On the other hand, the angular and spectral distribution of the radiation, for a high energy electron and in the forward direction along the tangent to the orbit, was given by Schwinger in a classic paper on synchrotron radiation (Schwinger 1949):

$$\frac{\mathrm{d}P_n}{\mathrm{d}\Omega}=\frac{e^2n^2}{3\pi^2a}(\gamma^{-2}+\psi^2)^2\left(K_{2/3}^2(\eta)+\frac{\psi^2}{\gamma^{-2}+\psi^2}K_{1/3}^2(\eta)\right) \tag{1.2}$$

where γ stands for the usual relativistic factor $\gamma=(1-v^2/c^2)^{-1/2}$, ψ is the angle between the line of observation and its projection on the orbit plane tangent to the orbital ring (see figure 1) and $\eta=(n\omega a/3c)(\gamma^{-2}+\psi^2)^{3/2}$. Experimental studies of synchrotron radiation have been carried out mainly in the forward direction and the results agree very satisfactorily with those predicted by equation (1.2). There have been numerous

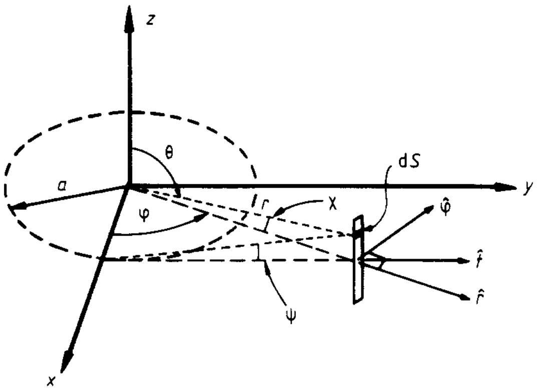

Figure 1. Coordinate system related to the measurement of the energy flux in the tangential direction $\hat{t}$. The slit is orthogonal to the unit vector $\hat{t}$.

experimental studies of synchrotron radiation (Tomboulian and Hartman 1956, Codling and Madden 1965, Haensel and Kunz 1967, Lemke and Labs 1967); a detailed list of references can be found in Codling's paper (Codling 1973).

The radial intensity of radiation, described by (1.1), as well as the one associated with the forward direction, given by (1.2), are independent of the distance between the detection point and the electron orbit. This distance independence follows from some approximations that can be easily justified far from the orbit. However, it is not *a priori* obvious that these approximations continue to hold near the electron orbit, because in this case the neglected pieces are difficult to estimate.

Schott's formula (1.1) is reproduced in almost any textbook on classical electrodynamics (see, for example, Landau and Lifshitz 1975). Its derivation is based on approximations that arise when the distance r between the detection point and the orbit centre is much larger than the orbit radius a, independent of the electron energy. For distances of the same order as the orbit radius the approximations are doubtful and, therefore, at such distances the validity of (1.1) is uncertain. The high energy form of this equation is formally identical to Schwinger's equation (1.2), but with the angle ψ replaced by $\chi = \frac{1}{2}\pi - \theta$ (Sokolov and Ternov 1968).

Unlike in the derivation of (1.1), the energy of the electron plays an essential role in the attainment of (1.2). For a high energy electron the radiation that reaches the slit, orthogonal to the tangential direction, is produced by a tiny arc of the electron orbit around the tangential point. Using this property, equation (1.2) follows easily (Jackson 1962) for distances, measured from the tangential point, that are much larger than the length of the tiny arc, which is usually the case in experiments; however this procedure can be hardly justified for detection near the tangential point.

The purpose of this paper is to study the radiation near the orbit, without using the abovementioned approximations. In this case the analytical study of the radiation is very complicated and for this reason we use this procedure only in the orbit plane. Furthermore, we are going to study the total intensity of the radiation, i.e. irrespective of frequency. Of course, our main interest is the radiation of a high energy electron. The most direct way for investigating the distance dependence of the radiation is by means of a power series in the parameter $\xi = a/r$. We use here precisely this method, which turns out to be convenient and fairly simple for studying the total intensity of the radiation in the orbit plane.

If $\boldsymbol{S}$ denotes the Poynting vector, T the electron period and $\hat{\boldsymbol{r}}=(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, the unit vector along the radial direction, then we will write

$$\langle\boldsymbol{S}\cdot\hat{\boldsymbol{r}}\rangle\equiv\int_{t=0}^{t=T}(\boldsymbol{S}\cdot\hat{\boldsymbol{r}})\,\mathrm{d}t \tag{1.3}$$

as a power series in the parameter $\xi=a/r$, namely

$$\langle\boldsymbol{S}\cdot\hat{\boldsymbol{r}}\rangle=\sum_{n=2}^{\infty}a_n\xi^n \tag{1.4}$$

with an analogous representation for the time integrated flux into the tangential direction $\boldsymbol{t}$ (see figure 1).

The analysis of the coefficients of the series representation (1.4) allows us to show that, for a high energy electron, the intensity of the radiation in the radial direction near the orbit differs strongly from the value given by Schott's formula. In fact, equation (1.1) gives, for the intensity of radiation of a high energy electron in the orbit plane, the value $7e^2\gamma^5/16a$, which is independent of the distance to the orbital ring. On the other hand, we find that the correct result is $(7e^2\gamma^5/16a)(1-\xi^2)^{-1/2}$. In order to investigate in more detail this increase in the radiation intensity as we move closer to the orbit, we use numerical techniques for the study of the radiation above and below the orbital plane. We find that the angular spread of the radiation decreases as we approach the electron orbit. This focusing effect becomes remarkable when the radiation is detected almost touching the electron orbit. In this case all the radiation emitted by the electron is focused into a line giving rise to a very high density of energy per unit area.

We also show, using analytical methods, that in the orbit plane the intensity of the radiation in the forward direction is given by Schwinger's formula and is independent of the distance to the tangential point. Strictly speaking, we have shown this distance independence only when we are not very close to the tangential point. In the tangential direction we cannot bring a detector to an arbitrary small distance from the tangential point. This difficulty does not exist, at least in principle, for detection along the radial direction.

2. The energy flux in the radial direction

In this section we are going to present first a simple proof that only the a_2 term of (1.4) gives rise to a global average flux through a spherical surface which encloses the region where the electron moves. Let us consider two spherical surfaces $\Sigma_{r'}$ and Σ_r centred at the orbit centre, of radii r' and r respectively, and such that $r'>r>a$. Let $\mathscr{D}$ be the volume limited by Σ_r and $\Sigma_{r'}$. Then the following conservation equation is satisfied in $\mathscr{D}$:

$$\nabla\cdot\boldsymbol{S}+\partial u/\partial t=0 \tag{2.1}$$

where $u=(\boldsymbol{E}^2+\boldsymbol{B}^2)/8\pi$ is the energy density of the electromagnetic field. From (2.1) we obtain

$$\int_{\Sigma_{r'}}\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{\Sigma}_{r'}-\int_{\Sigma_r}\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{\Sigma}_r=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{D}}u\,\mathrm{d}^3x. \tag{2.2}$$

No matter how complicated the electromagnetic field, it is periodic in time, due to the periodic nature of the electron movement. In particular, the integral over the volume $\mathscr{D}$ in (2.2) defines a periodic function of time. Therefore, if we integrate (2.2) over a period T we obtain

$$\int_{\Sigma_r} \langle \boldsymbol{S} \cdot \hat{\boldsymbol{r}} \rangle \, d\Sigma_r = \int_{\Sigma_{r'}} \langle \boldsymbol{S} \cdot \hat{\boldsymbol{r}} \rangle \, d\Sigma_{r'}. \tag{2.3}$$

Introducing here the expansion (1.4) and taking the limit $r' \to \infty$ we obtain

$$\sum_{n=3}^{\infty} \xi^n \int_{\Sigma_r} a_n \, d\Sigma_r = 0 \tag{2.4}$$

and because of the arbitrariness of r, it follows that

$$\int_0^{2\pi} \int_0^{\pi} a_n \sin\theta \, d\theta \, d\varphi = 0 \qquad \text{for } n \neq 2. \tag{2.5}$$

The electromagnetic field for an electron in arbitrary motion is given by (Jackson 1962)

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}, t) &= e\left(\frac{(\hat{\boldsymbol{n}} - \boldsymbol{\beta})}{\kappa^3 R^2} \gamma^{-2}\right) + \frac{e}{c}\left(\frac{\hat{\boldsymbol{n}} \times \{(\hat{\boldsymbol{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{\kappa^3 R}\right) \\ \boldsymbol{B}(\boldsymbol{x}, t) &= \hat{\boldsymbol{n}} \times \boldsymbol{E} \end{aligned} \tag{2.6}$$

where $\hat{\boldsymbol{n}}$ is the unit vector that points from the retarded electron position $z(t')$ to the detection point $\boldsymbol{x}$, $\boldsymbol{v}$ is the electron velocity dz/dt', $\boldsymbol{\beta} = \boldsymbol{v}/c$ and $R = |\boldsymbol{x} - z(t')|$ denotes the distance between the detection point and the retarded electron position. The right-hand side of (2.6) must be evaluated at the retarded time t', defined implicitly by $t = t' + |\boldsymbol{x} - z(t')|/c$, where t is the detection time. Substituting the fields (2.6) into the Poynting vector $\boldsymbol{S} = (c/4\pi)\boldsymbol{E} \times \boldsymbol{B}$ and using the circular motion condition $\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} = 0$, we find that $\boldsymbol{S}$ can be written as

$$\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 \tag{2.7}$$

where

$$\boldsymbol{S}_1 = \frac{e^2}{4\pi c \kappa^6 R^2} [\kappa^2 \dot{\boldsymbol{\beta}}^2 - \gamma^{-2} (\hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{\beta}})^2] \hat{\boldsymbol{n}} \tag{2.8}$$

which is built up from the acceleration parts of the electric and magnetic field, i.e. by means of the terms with R^{-1} dependence in (2.6). In spite of the fact that the piece $\boldsymbol{S}_2$ makes a contribution to the energy flux (1.3), it turns out that, for a high energy electron, it is negligible with respect to that of $\boldsymbol{S}_1$. In fact, in this case, following exactly the same procedure that we present here for the treatment of $\boldsymbol{S}_1$, it can be shown that the coefficients of the power series associated with $\langle \boldsymbol{S}_2 \cdot \hat{\boldsymbol{r}} \rangle$ are proportional to γ^3, in contrast with those of $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ which are proportional to γ^5. Then, since we are mainly interested in high energy electrons, in what follows we shall ignore the contribution to the energy flux due to the $\boldsymbol{S}_2$ term of the Poynting vector. Let us remark, however, that the property (2.5) remains valid separately for the coefficients of $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ as well as for those of $\langle \boldsymbol{S}_2 \cdot \hat{\boldsymbol{r}} \rangle$. This happens because $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ satisfy conservation equations of the type (2.1) (Teitelboim *et al* 1980). In particular, the covariant version of the conservation equation associated with $\boldsymbol{S}_1$ gives a clear picture of the radiation emitted by an electron, allowing a transparent visualisation of Rohrlich's local radiation criterion (Rohrlich 1965).

In our system of coordinates, which is centred at the orbit centre, let $\boldsymbol{x} = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ be the detection point and $\boldsymbol{z}(t') = a(\cos\omega t', \sin\omega t', 0)$ be the electron position in the XY plane, then

$$R \equiv |\boldsymbol{x} - \boldsymbol{z}(t')| = [r^2 - 2ar\sin\theta\cos(\omega t' - \varphi) + a^2]^{1/2} \tag{2.9}$$

and

$$\kappa \equiv 1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta} = 1 + (r/R)\beta\sin\theta\sin(\omega t' - \varphi) \tag{2.10}$$

where $\beta = a\omega/c$. Instead of working with R it is more convenient to introduce the dimensionless quantity ρ defined by

$$\rho \equiv [1 - 2\xi\sin\theta\cos(\omega t' - \varphi) + \xi^2]^{1/2} = R/r. \tag{2.11}$$

From equations (2.8)-(2.10) it is easy to obtain the following expression for the radial component of the energy flux associated with $\boldsymbol{S}_1$:

$$\boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} = \frac{e^2 c\beta^4}{32\pi a^4}\left(-\frac{\gamma^{-2}}{\kappa^6\rho^{-1}} + \frac{\gamma^{-2}}{\kappa^6\rho} + \frac{\gamma^{-2}}{\kappa^6\rho^3} - \frac{\gamma^{-2}}{\kappa^6\rho^5} - \frac{\gamma^{-2}\xi^2}{\kappa^6\rho} + \frac{4\xi^2}{\kappa^4\rho} - \frac{2\gamma^{-2}\xi^2}{\kappa^6\rho^3} \right.$$
$$\left. + \frac{4\xi^2}{\kappa^4\rho^3} + \frac{3\gamma^{-2}\xi^2}{\kappa^6\rho^5} + \frac{\gamma^{-2}\xi^4}{\kappa^6\rho^3} - \frac{4\xi^4}{\kappa^4\rho^3} - \frac{3\gamma^{-2}\xi^4}{\kappa^6\rho^5} + \frac{\gamma^{-2}\xi^6}{\kappa^6\rho^5} \right) \tag{2.12}$$

where κ and ρ are, of course, evaluated at the retarded time t'. By making the standard change of variables $\mathrm{d}t = \kappa\,\mathrm{d}t'$, and using the periodicity of $\boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}}$, we find that the radial energy flux during a period of the motion $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ can be written as

$$\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle = \int_{t'=0}^{t'=T} \kappa \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}}\,\mathrm{d}t' \tag{2.13}$$

where the integrand depends only on the time t'. Thus we conclude from (2.12) that the calculation of (2.13) is reduced to computing the following integrals:

$$F^{(p,q)} = \frac{1}{T}\int_0^T \frac{\mathrm{d}t'}{\kappa^p\rho^q} \tag{2.14}$$

with $p = 3, 5$ and $q = -1, 1, 3, 5$. The evaluation of $\langle \boldsymbol{S}_2 \cdot \hat{\boldsymbol{r}} \rangle$ can be carried out also by means of the integrals (2.14) but with $p = 3, 4, 5$ and $q = 1, 3, 5$. Due to the periodicity of the integrands in (2.14), it follows that these integrals are independent of the angle φ.

We will evaluate the integrals (2.14) by expanding them in a power series on the ξ parameter. Equation (2.14) can be written as

$$F^{(p,q)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}x}{\kappa^p\rho^q} \tag{2.15}$$

where $\kappa = 1 + \beta\sin\theta\sin x/\rho$ and $\rho = (1 - 2\xi\sin\theta\cos x + \xi^2)^{1/2}$. Now if we call $a_k^{(p)}$ the numbers

$$a_k^{(p)} = \frac{(-1)^k}{(p-1)!}\frac{(k+p-1)!}{k!} \tag{2.16}$$

corresponding to the power series expansion of the $(1+x)^{-p}$ function, then the $F^{(p,q)}$ function can be written as

$$F^{(p,q)} = \sum_{k=0}^{\infty} a_k^{(p)} \frac{1}{2\pi}\int_0^{2\pi} \frac{(\beta\sin\theta\sin x)^k\,\mathrm{d}x}{(1 - 2\xi\sin\theta\cos x + \xi^2)^{(k+q)/2}}. \tag{2.17}$$

The mathematical derivation of the above formula (and some others that we will find later) can be justified rigorously under the $\beta < 1$ and $\xi < 1$ hypothesis. It is easy to see that the integral (2.17) is null if k is odd. Then

$$F^{(p,q)} = \sum_{n=0}^{\infty} a_{2n}^{(p)} (\beta \sin \theta)^{2n} \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^{2n} x \, dx}{(1 - 2\xi \sin \theta \cos x + \xi^2)^{(2n+q)/2}}. \tag{2.18}$$

To calculate these integrals we make use of the following expansion (Gradsteyn and Ryzhik 1965):

$$(1 - 2\xi \sin \theta \cos x + \xi^2)^{-(2n+q)/2} = \sum_{m=0}^{\infty} \xi^m C_m^{(2n+q)/2}(\sin \theta \cos x) \tag{2.19}$$

where the $C_m^\lambda(t)$ are the Gegenbauer polynomials. Then we may write (2.18) as

$$F^{(p,q)} = \sum_{m=0}^{\infty} \xi^m \sum_{n=0}^{\infty} a_{2n}^{(p)} (\beta \sin \theta)^{2n} A_{2n,m}^{(q)} \tag{2.20}$$

where the $A_{2n,m}^{(q)}$ are given by

$$A_{2n,m}^{(q)}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \sin^{2n} x C_m^{(2n+q)/2}(\sin \theta \cos x) \, dx. \tag{2.21}$$

Due to the property $C_m^\lambda(-t) = (-1)^m C_m^\lambda(t)$, the integral (2.21) vanishes if the m index is odd. Thus, only the $A_{2n,2m}^{(q)}$ are different from zero. Introducing the $B_{2m}^{(p,q)}$ functions defined by

$$B_{2m}^{(p,q)} \equiv \sum_{n=0}^{\infty} a_{2n}^{(p)} (\beta \sin \theta)^{2n} A_{2n,2m}^{(q)} \tag{2.22}$$

we can write the $F^{(p,q)}$ functions as follows:

$$F^{(p,q)} = \sum_{m=0}^{\infty} \xi^{2m} B_{2m}^{(p,q)}. \tag{2.23}$$

If we introduce (2.23) in the time integrated energy flux $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$, we get the following expansion in terms of the ξ parameter:

$$\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle = \frac{e^2 \beta^3}{16 a^3} \sum_{m=1}^{\infty} b_{2m} \xi^{2m} \tag{2.24}$$

with

$$b_2 = \gamma^{-2}[-B_2^{(5,5)} + B_2^{(5,3)} + B_2^{(5,1)} - B_2^{(5,-1)}] + 8 B_0^{(3,1)} \tag{2.25}$$

$$\begin{aligned} b_4 = \gamma^{-2}[&-B_4^{(5,5)} + B_4^{(5,3)} + B_4^{(5,1)} - B_4^{(5,-1)} + 3B_2^{(5,5)} - 2B_2^{(5,3)} \\ &- B_2^{(5,1)} - 2B_0^{(5,3)}] + 4[B_2^{(3,3)} + B_2^{(3,1)} - B_0^{(3,3)}]. \end{aligned} \tag{2.26}$$

Furthermore, in general for $m \geq 3$ the coefficients b_{2m} are given by

$$\begin{aligned} b_{2m} = \gamma^{-2}[&-B_{2m}^{(5,5)} + B_{2m}^{(5,3)} + B_{2m}^{(5,1)} - B_{2m}^{(5,-1)} + 3B_{2m-2}^{(5,5)} - 2B_{2m-2}^{(5,3)} - B_{2m-2}^{(5,1)} \\ &- 3B_{2m-4}^{(5,5)} + B_{2m-4}^{(5,3)} + B_{2m-6}^{(5,5)}] + 4[B_{2m-2}^{(3,3)} + B_{2m-2}^{(3,1)} - B_{2m-4}^{(3,3)}]. \end{aligned} \tag{2.27}$$

The relevant point of our procedure lies in the fact that the functions $B_{2m}^{(p,q)}$ that appear in the coefficients b_{2m} can be exactly evaluated in an explicit way for arbitrary values of the angle θ, electron energy and index m. Let us consider first the $A_{2n,2m}^{(q)}$

given by equation (2.21). Gegenbauer's polynomials $C^{\lambda}_{2m}(t)$ can be written as (Gradsteyn and Ryzhik 1965)

$$C^{\lambda}_{2m}(t)=\frac{(-1)^m F(-m;\, m+\lambda;\, \tfrac{1}{2};\, t^2)}{(\lambda+m)B(\lambda,\, m+1)} \tag{2.28}$$

or, in a more explicit way,

$$C^{\lambda}_{2m}(t)=\frac{(-1)^m}{\Gamma(\lambda)\Gamma(m+1)}\sum_{k=0}^{m}\binom{m}{k}\frac{2^k(-1)^k}{(2k-1)!!}\Gamma(m+k+\lambda)t^{2k} \tag{2.29}$$

where we have used the convention $(-1)!!=1$. Introducing this expression in (2.21), and with the help of the result

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^{2n}x\cos^{2k}x\,\mathrm{d}x=\frac{(2k-1)!!\,(2n-1)!!}{2^{n+k}\Gamma(n+k+1)} \tag{2.30}$$

we find

$$A^{(q)}_{2n,2m}(\theta)=\frac{(-1)^m(2n-1)!!}{\Gamma(n+q/2)\Gamma(m+1)2^n}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{\Gamma(m+k+n+q/2)}{\Gamma(n+k+1)}\sin^{2k}\theta. \tag{2.31}$$

Since in this paper $q\geqslant -1$, equation (2.31) becomes ambiguous only if $m=n=q=0$. However, in this case it is easy to see that $A^{(0)}_{0,0}=1$.

The $B^{(p,q)}_{2m}$ functions defined in (2.22) are essentially polynomials in the variable

$$X=(1-\beta^2\sin^2\theta)^{-1/2}. \tag{2.32}$$

Although the calculation of $B^{(p,q)}_{2m}$ is direct, it becomes quite wearisome for high values of m. As an illustration of the method for evaluating the $B^{(p,q)}_{2m}$, let us consider the particular case of $B^{(5,q)}_0$. From (2.31) it follows that $A^{(q)}_{2n,0}=(2n-1)!!/(2n)!!$, then

$$B^{(5,q)}_0=\sum_{n=0}^{\infty}a^{(5)}_{2n}\frac{(2n-1)!!}{(2n)!!}(\beta\sin\theta)^{2n}. \tag{2.33}$$

Replacing the value for $a^{(5)}_{2n}$ obtained from (2.16) we obtain

$$B^{(5,q)}_0=\frac{1}{4!}\sum_{n=0}^{\infty}[(2n+7)!!-6(2n+5)!!+3(2n+3)!!]\frac{(\beta\sin\theta)^{2n}}{(2n)!!}. \tag{2.34}$$

These series are related to the X function defined in (2.32) since

$$X^{2k+1}=\frac{1}{(2k-1)!!}\sum_{n=0}^{\infty}\frac{(2n+2k-1)!!}{(2n)!!}(\beta\sin\theta)^{2n}. \tag{2.35}$$

From this we conclude that $B^{(5,q)}_0$ is given by

$$B^{(5,q)}_0=\tfrac{1}{8}(35X^9-30X^7+3X^5). \tag{2.36}$$

The evaluation of any $B^{(p,q)}_{2m}$ can be carried out following the same steps used in $B^{(5,q)}_0$. However, in the computation of the coefficients b_{2m} of (2.24) it is not convenient to deal with each $B^{(p,q)}_{2m}$ separately, as can be seen from the fact that the $B^{(5,q)}_{2m}$ include spurious terms of powers higher than γ^5 for a high energy electron, as is shown in (2.36). Actually, strong simplifications arise if we evaluate together the set of terms $B^{(p,q)}_{2m}$ that have the same index p in b_{2m}. For example, after some algebra we get the following exact expressions for the first three coefficients of (2.24):

$$b_2=(1/\beta^2)[-5\gamma^{-2}X^7+6(1+\beta^2)X^5-(1+3\beta^2)X^3] \tag{2.37}$$

$$b_4 = (1/2\beta^4)[35(2\beta^2-3)\gamma^{-4}X^9 + 30(8-7\beta^2+2\beta^4)\gamma^{-2}X^7 - 3(55-60\beta^2+23\beta^4-2\beta^6)X^5 + (30-10\beta^2+4\beta^4)X^3] \tag{2.38}$$

$$b_6 = (1/8\beta^6)[315(-5+4\beta^2)\gamma^{-6}X^{11} + 70(65-73\beta^2+20\beta^4)\gamma^{-4}X^9 + 75(-57+82\beta^2-37\beta^4+4\beta^6)\gamma^{-2}X^7 + 12(105-182\beta^2+105\beta^4-20\beta^6)X^5 + 40-16\beta^2]. \tag{2.39}$$

The coefficient b_2 is well known (Landau and Lifshitz 1975). These equations show that the complexity of the b_{2m} increases rapidly with the index m. Fortunately, in the orbit plane, where for a high energy electron most of the radiation is concentrated, we can evaluate the b_{2m} in an exact way for any index m. This simplification arises because the functions $A^{(q)}_{2n,2m}$ defined in (2.21) satisfy a simple recurrence relation when $\theta = \pi/2$. In fact from Watson (1952)

$$\int_0^\pi C^\mu_{n+1}(\cos\varphi)(\sin\varphi)^{2\nu}\,d\varphi = \frac{(n+2\mu-2\nu-1)(n+2\mu-1)}{(2\nu+n+1)(n+1)}\int_0^\pi C^\mu_{n-1}(\cos\varphi)(\sin\varphi)^{2\nu}\,d\varphi \tag{2.40}$$

it follows that

$$A^{(q)}_{2n,2m}(\pi/2) = \frac{(2m+q-2)(2m+2n+q-2)}{2m(2m+2n)}A^{(q)}_{2n,2(m-1)}(\pi/2). \tag{2.41}$$

Using this relation in (2.27) we obtain

$$b_{2m} = \frac{(2m-3)!!}{3(2m-2)!!}\beta^{-2m}\{21\gamma^5 - 6(3m+2)\gamma^3 + (2m-1)(2m+7)\gamma - (2m-1)(2m-2)(2m-3)\gamma^{-1} + \gamma^{-2}H(m,5) - 2[6+(m+2)\gamma^{-2}]H(m,3) + [12(2m+2)-(2m-1)(2m-5)\gamma^{-2}]H(m,1) - (2m-1)[12-(2m-2)(2m-3)\gamma^{-2}]H(m,-1)\} \tag{2.42}$$

where

$$H(m,2k-1) \equiv \sum_{r=0}^{m-1}\frac{(2r+2k-1)!!}{(2r)!!}\beta^{2r}. \tag{2.43}$$

Equation (2.42) is considerably simplified for a high energy electron. In order to obtain its high energy form let us consider the quantities $H(m,2k-1)$. They satisfy

$$H(m,2k-1) < \sum_{r=0}^{m-1}\frac{(2r+2k-1)!!}{(2r)!!} < \sum_{r=0}^{m-1}\frac{(2r+2k)!!}{(2r)!!} = (2k)!!\binom{k+m}{k+1}.$$

Thus, when m increases we have the rough estimate $H(m, 2k-1) = O(m^{k+1})$, which is enough for our purpose. Now by neglecting quantities of order γ^4 in front of γ^5, we get from (2.42) the following high energy form $\tilde{b}_{2m}$ of the coefficients b_{2m} in the orbit plane:

$$\tilde{b}_{2m} = \frac{7(2m-3)!!}{(2m-2)!!}\gamma^5. \tag{2.44}$$

This equation is valid for $3 \leq m < \gamma$. On the other hand, from (2.25) and (2.26) it follows that the relativistic value of b_2 and b_4 in the orbit plane are given by $\tilde{b}_2 = 7\gamma^5$

and $\tilde{b}_4 = 7\gamma^5/2$. These results allow us to study $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ in the orbit plane as a function of the distance to the electron orbit. At distances close to the orbit we need a large number of terms in the power series (2.24). If ξ is not too close to 1, it is enough to take into account the terms with $m < \gamma$. Then for a high energy electron we have

$$\begin{aligned}\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle &= \frac{e^2}{16a^3} \sum_{m=1}^{m<\gamma} \tilde{b}_{2m} \xi^{2m} \\ &= \frac{7}{16}\left(\frac{e^2}{a^3}\right)\gamma^5 \xi^2 \left(1 + \tfrac{1}{2}\xi^2 + \ldots + \frac{(2m-3)!!}{(2m-2)!!}\xi^{2m-2} + \ldots\right) \\ &= \frac{7}{16}\left(\frac{e^2}{a^3}\right)\gamma^5 \xi^2 (1-\xi^2)^{-1/2}. \end{aligned} \tag{2.45}$$

Let $\mathrm{d}\Omega'$ be an element of solid angle around the orbit plane with apex at the orbit centre; then the corresponding surface element at a distance r is $\mathrm{d}S = a^2 \xi^{-2}\,\mathrm{d}\Omega'$. Therefore the radial energy flux in the orbit plane is given by

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega'} = \frac{7}{16}\left(\frac{e^2}{a}\right)\gamma^5 (1-\xi^2)^{-1/2}. \tag{2.46}$$

This formula shows an important dependence of the energy flux with the distance to the electron orbit, in contrast with the result obtained from Schott's formula (1.1), which is $\frac{7}{16}(e^2/a)\gamma^5$, irrespective of distance. The standard derivation of Schott's formula (1.1) uses only the asymptotic form of the fields. These far fields fix the a_2 coefficient in (1.4); on the other hand, the factor $(1-\xi^2)^{-1/2}$ that appears in (2.46) arises from the terms a_n with $n \geqslant 4$. In general, the coefficients a_n with $n \geqslant 4$ make a contribution to the energy flux in a given direction, even though the energy flux associated with them across the whole surface of a sphere enclosing the electron orbit is zero.

Although the coefficient b_{2m} are positive in the orbit plane for any index m, they have a complicated oscillatory behaviour outside this plane, which is a function of the electron energy, the angle θ and the index m. In order to clarify this point, let us consider the high energy form of b_2, b_4 and b_6. If we introduce the angle χ measured with respect to the orbit plane, i.e. $\chi = \pi/2 - \theta$, it is easy to see that (2.32) can be

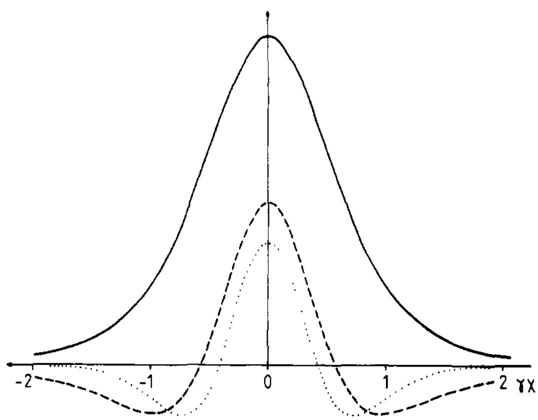

Figure 2. Curves related to the radial energy flux in the relativistic limit (——, $\tilde{b}_2$; - - -, $\tilde{b}_4$; · · ·, $\tilde{b}_6$).

written approximately as $X=\gamma Z$, where Z is defined by $Z=[1+(\gamma\chi)^2]^{-1/2}$. Then from (2.37)-(2.39) we get

$$\begin{aligned}\tilde{b}_2 &= \gamma^5(-5Z^2+12)Z^5\\ \tilde{b}_4 &= \tfrac{1}{2}\gamma^5(-35Z^4+90Z^2-48)Z^5\\ \tilde{b}_6 &= \tfrac{1}{8}\gamma^5(-315Z^6+840Z^4-600Z^2+96)Z^5.\end{aligned} \tag{2.47}$$

In figure 2 we show the $\tilde{b}_2$, $\tilde{b}_4$ and $\tilde{b}_6$ curves without an overall factor proportional to γ^5. The angle χ is measured in γ^{-1} units. The increasing complexity of the $\tilde{b}_{2m}$ with m makes it very difficult to obtain an analytical expression, for angles other than $\theta=\pi/2$, for the radial radiation intensity.

3. The energy flux in the forward direction

In this section we study the distance dependence, if there is any, of the energy flux in the forward direction and along the tangent to the electron orbit. Our considerations will be restricted to the orbit plane (using numerical techniques, Risley *et al* (1982) have studied the forward energy flux around the orbit plane). Let $\hat{t}$ be the unit vector into the tangential direction to the electron orbit, as shown in figure 1. Then we have

$$\hat{t}=\hat{r}\sin\varphi+\hat{\varphi}\cos\varphi \tag{3.1}$$

where $\hat{\varphi}$ is the unit vector $(-\sin\varphi, \cos\varphi, 0)$. In particular, the time integrated energy flux in the forward direction is given by

$$\langle S_1\cdot\hat{t}\rangle=\langle S_1\cdot\hat{r}\rangle\sin\varphi+\langle S_1\cdot\hat{\varphi}\rangle\cos\varphi. \tag{3.2}$$

As we have already calculated $\langle S_1\cdot\hat{r}\rangle$ in (2.45), it remains to specify $\langle S_1\cdot\hat{\varphi}\rangle$. Now from (2.8) it follows that

$$\begin{aligned}S_1\cdot\hat{\varphi}=\frac{e^2c\beta^3\xi}{16\pi a^4}\bigg[4\xi^2\left(\frac{1}{\kappa^4\rho^2}-\frac{1}{\kappa^3\rho^2}\right)+\gamma^{-2}\bigg(\frac{1}{\kappa^5}-\frac{1}{\kappa^6}+\frac{1}{\kappa^5\rho^4}-\frac{1}{\kappa^6\rho^4}+\frac{2}{\kappa^6\rho^2}\\ -\frac{2}{\kappa^5\rho^2}+\frac{2\xi^2}{\kappa^5\rho^2}-\frac{2\xi^2}{\kappa^6\rho^2}+\frac{2\xi^2}{\kappa^6\rho^4}-\frac{2\xi^2}{\kappa^5\rho^4}+\frac{\xi^4}{\kappa^5\rho^4}-\frac{\xi^4}{\kappa^6\rho^4}\bigg)\bigg].\end{aligned} \tag{3.3}$$

Therefore, the time integrated energy flux $\langle S_1\cdot\hat{\varphi}\rangle$ can be expressed in powers of the ξ parameter, with coefficients which are combinations of the $B_{2m}^{(p,q)}$ functions defined in equation (2.22). From (3.3) we get

$$\langle S_1\cdot\hat{\varphi}\rangle=\frac{e^2\beta^2\xi}{8a^3}\sum_{m=1}^{\infty}c_{2m}\xi^{2m} \tag{3.4}$$

where

$$\begin{aligned}c_2 &= \gamma^{-2}(-B_2^{(5,4)}+2B_2^{(5,2)}-B_2^{(5,0)}+2B_0^{(5,4)}-2B_0^{(5,2)})\\ &\quad+\gamma^{-2}(B_2^{(4,4)}-2B_2^{(4,2)}+B_2^{(4,0)}-2B_0^{(4,4)}+2B_0^{(4,2)})+4B_0^{(3,2)}-4B_0^{(2,2)}\end{aligned} \tag{3.5}$$

and in general for $m\geqslant 2$

$$\begin{aligned}c_{2m} &= \gamma^{-2}(-B_{2m}^{(5,4)}+2B_{2m}^{(5,2)}-B_{2m}^{(5,0)}+2B_{2(m-1)}^{(5,4)}-2B_{2(m-1)}^{(5,2)}-B_{2(m-2)}^{(5,4)})\\ &\quad+\gamma^{-2}(B_{2m}^{(4,4)}-2B_{2m}^{(4,2)}+B_{2m}^{(4,0)}-2B_{2(m-1)}^{(4,4)}+2B_{2(m-1)}^{(4,2)}+B_{2(m-2)}^{(4,4)})\\ &\quad+4B_{2(m-1)}^{(3,2)}-4B_{2(m-1)}^{(2,2)}.\end{aligned} \tag{3.6}$$

In the orbit plane the coefficients (3.5) and (3.6) are very simple. In fact, using the recurrence relation (2.41), we find that they are given by

$$c_{2m} = \tfrac{7}{2}(\gamma^5 - \gamma^3) \tag{3.7}$$

which, unlike the coefficients of the power series associated with $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$, are independent of the index m. From (3.7) we get the following exact result for $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{\varphi}} \rangle$ in the orbit plane:

$$\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{\varphi}} \rangle = \frac{7}{16} \frac{e^2 \beta^2}{a^3} (\gamma^5 - \gamma^3) \xi^3 (1 - \xi^2)^{-1} \tag{3.8}$$

which for a high energy electron is reduced to

$$\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{\varphi}} \rangle = \frac{7}{16} \left(\frac{e^2}{a^3} \right) \gamma^5 \xi^3 (1 - \xi^2)^{-1}. \tag{3.9}$$

Introducing this result, together with the value of $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ given by (2.45) in equation (3.2), and using the fact that $\cos \varphi = \xi$, we obtain

$$\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{t}} \rangle = \frac{7}{16} \left(\frac{e^2}{a^3} \right) \gamma^5 \xi^2 (1 - \xi^2)^{-1}. \tag{3.10}$$

The surface element in the slit, around the orbit plane, can be written as $\mathrm{d}S = a^2 \xi^{-2} (1 - \xi^2)\, \mathrm{d}\Omega$, where $\mathrm{d}\Omega$ is the solid angle associated with the surface element $\mathrm{d}S$ as is seen from the tangential point. Then, from equation (3.10) it follows that the forward energy flux in the orbit plane is

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{7}{16} \left(\frac{e^2}{a} \right) \gamma^5. \tag{3.11}$$

Therefore, when the radiation is measured along the tangent to the orbit we get the value $7e^2\gamma^5/16a$ irrespective of distance. This distance independence of the radiation intensity along the tangent deserves some comments. For detection near the tangential point, the element of solid angle $\mathrm{d}\Omega$ must be defined in a precise way. In this case the exact location of the tangential point turns out to be very important, but this is not an easy matter because the orbit looks like a straight line for observation close to it. Besides, the formula $a^2 \xi^{-2} (1 - \xi^2)\, \mathrm{d}\Omega$ for the surface element is inaccurate in the neighbourhood of the tangential point. A more fundamental difficulty is associated with the process of radiation detection itself. In fact, if the detector is brought near the tangential point along the tangent, the angle $\delta\alpha$ that defines the angular width of the detector in the horizontal direction, i.e. in the orbit plane, is bisected by the tangent to the orbit. Therefore, the electron beam will collide with the detector, before it reaches the tangential point.

The above difficulties do not appear for detection in the radial direction, where we can, at least in principle, measure the radiation at arbitrary small distances from the electron orbit. In this case, according to (2.46), the radiation intensity differs considerably from the value obtained at large distances. In order to appreciate this effect let us consider, for instance, the case when the orbit radius is 1 m. Then if we detect radiation at 5 mm from the orbit, we have $\xi = 0.995$, i.e. $\xi^2 = 0.99$. For this value of ξ^2, equation (2.46) shows that the radiation intensity is one order of magnitude higher than $7e^2\gamma^5/16a$. At such distances it is easy to justify the validity of (2.46) for electron energies above 500 MeV. Indeed, in the power series representation of $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ we are considering close to one thousand terms and for $\xi^2 = 0.99$ we have $(\xi^2)^{1000} \sim 10^{-5}$. With the aim of getting a more detailed information about this effect, we will analyse the distance dependence of $\mathrm{d}I/\mathrm{d}\Omega'$ outside the orbit plane in the next section.

4. The energy flux outside the orbit plane

Due to the complicated oscillatory behaviour of the coefficients b_{2m} outside the orbit plane, the treatment of $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ by means of a power series in the parameter ξ is hopeless in this case. For this reason we will use numerical techniques for computing $\langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle$ outside the orbit plane. With the purpose of avoiding the spurious terms, with powers higher than γ^5, that appear in the computation of the functions $F^{(5,q)}$ defined by (2.15), it is convenient to rearrange the terms involved in equation (2.13), writing

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega'} \equiv \langle \boldsymbol{S}_1 \cdot \hat{\boldsymbol{r}} \rangle r^2 \tag{4.1}$$

as follows:

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega'} = \frac{7}{16}\left(\frac{e^2}{a}\right)\gamma^5 A \tag{4.2}$$

with

$$A = \frac{4\gamma^{-5}}{7\pi}\left(\int_0^{2\pi} \frac{(1-\xi \sin\theta \cos x)\,\mathrm{d}x}{\kappa^3 \rho^3} - \gamma^{-2}\int_0^{2\pi} \frac{(\sin\theta\cos x - \xi)^2(1-\xi\sin\theta\cos x)\,\mathrm{d}x}{\kappa^5\rho^5}\right). \tag{4.3}$$

Numerical analysis of the integrands of (4.3) shows that, for a high energy electron, they are important only in a small neighbourhood of the point $\cos x = \xi$ and $\sin x = -(1-\xi^2)^{1/2}$. The piece of the orbit, around this point, that contributes significantly to the integrals depends on the parameters ξ, β and θ. However, the length of this neighbourhood changes only a little with ξ, if ξ is not too close to one, as happens for the curves drawn in figure 3. This fact can be used in order to obtain an approximate analytical expression for the integral of (4.3). We will not pursue this procedure here. Instead, we perform numerical integration using a Monte Carlo program VEGAS (Lepage 1978), which can handle integrals like the ones that appear in equation (4.3) very well.

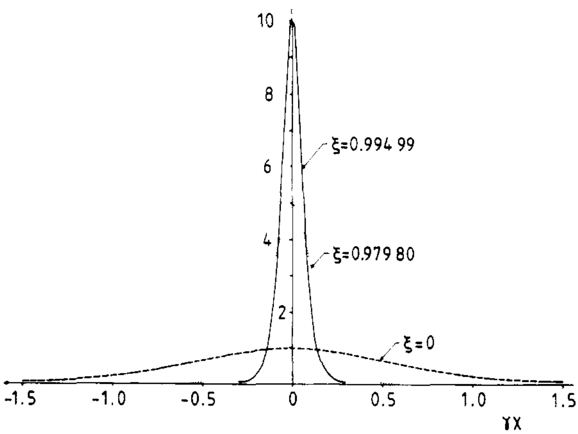

Figure 3. Plots of the radial intensity of radiation as a function of the elevation angle χ measured in γ^{-1} units for three different distances from the electron orbit. These curves show in a clear way the focusing effect of the radiation in the orbit plane, as we move closer to the electron orbit.

Figure 3 shows some plots of the amplitude A for different values of the distance to the electron orbit. The broadest curve corresponds to $\xi = 0$ and represents the standard Schott result; the others are associated with the values $\xi = 0.9798$ and $\xi = 0.994\,99$. The energy of the electron is the same for the three curves and has a value of 500 MeV. The angle χ is measured with respect to the orbit plane, i.e. $\chi = \pi/2 - \theta$.

Figure 3 shows a combination of two effects as we approach the electron orbit. Together with an increase of the radiation intensity, the angular spread of the radiation decreases. This focusing of the radiation in the orbit plane can be understood starting from equation (2.46) and using conservation of energy. Due to the fact that the $\boldsymbol{S}_1$ piece of the Poynting vector satisfies equation (2.1) (Teitelboim *et al* 1980), it can be shown, following the same procedure for obtaining (2.5), that the coefficients b_{2m} of (2.24) have the property

$$\int_0^{2\pi} \int_0^{\pi} b_{2m} \sin\theta \, d\theta \, d\varphi = 0 \tag{4.4}$$

for $m \geqslant 2$. Therefore, for $r > a$ (but otherwise arbitrary), we have

$$\int_0^{2\pi} \int_0^{\pi} \frac{dI}{d\Omega'} d\Omega' = \frac{e^2\beta^3\pi}{8a} \int_0^{\pi} b_2 \sin\theta \, d\theta = \frac{4\pi}{3}\left(\frac{e^2}{a}\right)\gamma^4 \tag{4.5}$$

where we have used (2.37) for a high energy electron. Now, introducing in (4.5) the expression (4.2) we obtain

$$\int_0^{\pi/2} A(\gamma, \chi, \xi) \, d\chi = \tfrac{16}{21}\gamma^{-1}. \tag{4.6}$$

Thus, independently of the value of ξ, the area under the curves of figure 3 must be the same. The focusing effect then follows from the fact that in the orbit plane the intensity of radiation increases as we approach the electron orbit, as is shown by equation (2.46). Numerical calculations show that for an electron of 500 MeV, the increasing behaviour $(1-\xi^2)^{-1/2}$ given in equation (2.46) is appropriate even for values of ξ as close to one as $1 - \xi = 10^{-6}$. For an idealised source of one monoenergetic electron in circular orbit, the focusing of the radiation in the orbit plane becomes a remarkable effect. Thus for a detector located almost touching the electron orbit, all the emitted radiation tends to be concentrated into a line, producing a very high density of energy per unit area.

Equation (4.6) can be written in a more compact way in terms of a new variable t defined by $\chi = t\gamma^{-1}$. In fact we have

$$\int_0^{\infty} A(\gamma, t\gamma^{-1}, \xi) \, dt = \tfrac{16}{21}. \tag{4.7}$$

More detailed information about the amplitude A that appears in this equation, as a function of the electron energy and the distance to the orbit, will be considered in a further paper.

In real machines the electron beam has a finite size, which certainly plays an important role for detection near the orbit. Starting from the incoherence of synchrotron radiation, we also expect an important dependence on the distance for the intensity of radiation in a real machine. It is rather obvious, however, that the finite size of the beam will prevent the focusing of the radiation into a line, as happens in the case of one electron. It seems clear that the increase of the intensity and focusing of the radiation, obtained when we approach the orbit, will be more significant for machines with small beam size. In any case, in order to have a very high intensity of radiation, it would be sufficient that the electron beam has a small cross section only in a tiny arc of the orbit located in the vicinity of the detector.

Here, we will not attempt to give a quantitative description of the influence of the beam size on the radiation intensity. The treatment of the effect of the beam size on the radiation is a very delicate matter, especially for detection near the orbit. Besides the complications for determining the cross section of the beam (Tomboulian and Hartman 1956, Codling and Madden 1965), there are questions about the distribution of the electrons in the orbit. Other important sources of difficulties are those associated with the coherence of the radiation near the orbit and the ones due to the oscillations of the electrons in their orbits. We hope to deal with these matters in the near future.

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